

Two-component soliton systems and the Painlevé equations

Mikio Murata

Department of Physics and Mathematics,
Aoyama Gakuin University,
5-10-1 Fuchinobe Sagamihara-shi,
Kanagawa 229-8558, Japan

Abstract

We give an extension of the two-component KP hierarchy by considering additional time variables. We obtain the linear 2×2 system by taking into consideration the hierarchy through a reduction procedure. The Lax pair of the Schlesinger system and the sixth Painlevé equation is given from this linear system. A unified approach to treat the other Painlevé equations from the usual two-component KP hierarchy is also considered.

Keywords: Painlevé equations, NLS-like equations, Completely integrable systems.

2000 Mathematics Subject Classification: 34M55, 35Q55, 37K10.

1 Introduction

In this paper, we deal with the Painlevé equations and the soliton systems, and also relations among them. Important fact we give attention is correspondence between the Painlevé equations and the holonomic deformation, that is, the monodromy preserving deformation of linear differential equations. On the other hand, the Kadomtsev-Petviashvili (KP) hierarchy arises from the isospectral deformation of the eigenvalue problem. The aim of this paper is to establish correspondence between the isospectral deformation and

the monodromy preserving deformation. We construct an extension of the two-component KP hierarchy by introducing new time variables. We give also the relation between this hierarchy and the sixth Painlevé equation. We show also the relation between the usual two-component KP hierarchy and the other Painlevé equations.

In this introduction, we begin by reviewing the theory of the Painlevé equations and the soliton theory. Then we state main results of the present article. In Section 2, we construct an extension of the two-component KP hierarchy by employing the Sato-Wilson formalism. In Section 3, we consider the holonomic deformation based on this extended hierarchy and obtain the nonlinear system that describes the condition of this deformation. We see that the nonlinear system reduces to P_{VI} . In Section 4, we study the holonomic deformation that contains the two-component KP hierarchy and show that the nonlinear systems that describes the condition of this deformation reduce to the other Painlevé equations, P_V , P_{IV} , P_{III} and P_{II} .

1.1 Painlevé equations

P. Painlevé studied second order ordinary differential equations of the form

$$\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right) \quad (1.1)$$

where F is analytic in t and rational in y and dy/dt . He tried to determine all of equations without a movable critical point. P. Painlevé and B. Gambier arrived at the following six equations, which are known as the Painlevé

equations ([5, 29]):

$$P_I: \frac{d^2 y}{dt^2} = 6y^2 + t, \quad (1.2)$$

$$P_{II}: \frac{d^2 y}{dt^2} = 2y^3 + ty + \alpha, \quad (1.3)$$

$$P_{III}: \frac{d^2 y}{dt^2} = \frac{1}{y} \left(\frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}, \quad (1.4)$$

$$P_{IV}: \frac{d^2 y}{dt^2} = \frac{1}{2y} \left(\frac{dy}{dt} \right)^2 + \frac{3}{2} y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \quad (1.5)$$

$$P_V: \frac{d^2 y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \left(\frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{(y-1)^2}{t} \left(\alpha y + \frac{\beta}{y} \right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \quad (1.6)$$

$$P_{VI}: \frac{d^2 y}{dt^2} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left(\frac{dy}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left[\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2} \right]. \quad (1.7)$$

The Painlevé equations also appear in the problem of the monodromy preserving deformation of linear differential equations. R. Fuchs ([4]) considered the second order linear differential equation of Fuchsian type:

$$\frac{d^2 \psi}{d\lambda^2} = p(\lambda, t) \psi \quad (1.8)$$

with the four regular singular points, $\lambda = 0, 1, \infty, t$ and the apparent singularity $\lambda = y$. He proved that the sixth Painlevé equation, P_{VI} , describes the condition that the linear differential equation has a fundamental system of solutions whose monodromy are independent of a variable t . Results obtained by R. Garnier ([7]) is concerned with the isomonodromic deformation of the second order linear differential equation with irregular singularities. He showed that the other five Painlevé equations, $P_I, P_{II}, P_{III}, P_{IV}, P_V$, are obtained from completely integrability conditions of extended systems of the linear differential equation. L. Schlesinger ([32]) considered the isomonodromic deformation of the linear system of the first order differential

equations with regular singularities:

$$\frac{d\Psi}{d\lambda} = \sum_{\nu=1}^n \frac{A_\nu}{\lambda - a_\nu} \Psi, \quad (1.9)$$

and obtained the system of nonlinear differential equations:

$$\frac{\partial A_\nu}{\partial a_\mu} = \frac{[A_\mu, A_\nu]}{a_\mu - a_\nu} \quad (\mu \neq \nu), \quad (1.10)$$

$$\frac{\partial A_\nu}{\partial a_\nu} = - \sum_{\kappa(\neq \nu)} \frac{[A_\kappa, A_\nu]}{a_\kappa - a_\nu}, \quad (1.11)$$

where

$$[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu.$$

This system is obtained from the complete integrability condition of the extended system of (1.9):

$$\frac{\partial \Psi}{\partial \lambda} = \sum_{\nu=1}^n \frac{A_\nu}{\lambda - a_\nu} \Psi, \quad (1.12)$$

$$\frac{\partial \Psi}{\partial a_\nu} = - \frac{A_\nu}{\lambda - a_\nu} \Psi. \quad (1.13)$$

M. Jimbo, T. Miwa and K. Ueno ([9]) established general theory of monodromy preserving deformation for the matrix system of first order linear ordinary differential equations with regular or irregular singularities:

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi, \quad (1.14)$$

where

$$A(\lambda) = \sum_{\nu=1}^n \sum_{k=1}^{r_\nu} \frac{A_{\nu,k}}{(\lambda - a_\nu)^k} - \sum_{k=2}^{r_\infty} A_{\infty,k} \lambda^{k-2}. \quad (1.15)$$

In [9] monodromy data are being considered as a set of Stokes multipliers, connection matrices and exponents of formal monodromy, and they define the generalized monodromy preserving deformation is the deformation as monodromy data of a fundamental system of solutions are preserved. M. Jimbo and T. Miwa ([10]) presented the linear systems with 2×2 matrices that the Painlevé equations are obtained from the compatibility condition of them.

This linear systems are called the Lax pairs for the Painlevé equations. When $A(\lambda)$ (1.15) has a pole of degree r_ν at $\lambda = a_\nu$, the equation (1.14) is said to have a singular point of Poincaré rank $r_\nu - 1$. We associate with each of $\lambda = a_\nu$ ($\nu = 1, \dots, n$) a natural number r_ν such that the Poincaré rank of $\lambda = a_\nu$ is given by $r_\nu - 1$. We also associate with $\lambda = \infty$ a natural number r_∞ such that the Poincaré rank of $\lambda = \infty$ is given by $r_\infty - 1$. Then we can represent such a system of linear differential equations by the following symbol:

$$(r_1, r_2, \dots, r_n, r_\infty). \quad (1.16)$$

The system of linear differential equations considered in the studies on the Schlesinger system (1.9) is of the type:

$$\underbrace{(1, 1, \dots, 1)}_{n+1}. \quad (1.17)$$

By the use of this notation, the correspondence of the types of the linear system with 2×2 matrices to the types of the Painlevé equations is the following:

$$P_{VI}: (1, 1, 1, 1), \quad (1.18a)$$

$$P_V: (1, 1, 2), \quad (1.18b)$$

$$P_{IV}: (1, 3), \quad (1.18c)$$

$$P_{III}: (2, 2), \quad (1.18d)$$

$$P_{II}: (4). \quad (1.18e)$$

1.2 Soliton systems

The soliton theory is based on studies of the Korteweg-de Vries (KdV) equation. N. J. Zabusky and M. D. Kruskal ([37]) studied the behavior of the numerical solutions of the KdV equation. They found that the solitary wave solutions had behavior similar to the superposition principle, despite the fact that the waves themselves were highly nonlinear. They named such waves solitons. This result led C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura ([6]) to the discovery of the inverse scattering transform method to solve the initial value problems for the KdV equation. P. D. Lax ([19]) showed that the KdV equation is equivalent to the isospectral integrability condition for pairs of linear operators, known as Lax pairs. If we introduce

the differential operators,

$$L = \partial_x^2 + u, \quad (1.19)$$

$$B = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}\partial_x u, \quad (1.20)$$

then the inverse scattering scheme for the KdV equation is written by

$$L\psi = \lambda\psi, \quad (1.21)$$

$$\partial_t\psi = B\psi. \quad (1.22)$$

If the eigenvalue λ is independent of x and t , then the compatibility condition of the equations (1.21) and (1.22) yields

$$\partial_t L = [B, L], \quad (1.23)$$

which reduces to the KdV equation. An extension of the Lax equation was given by V. E. Zakharov and A. B. Shabat ([38]). They treated the following equation for linear differential operators:

$$\partial_y B - \partial_t C + [B, C] = 0, \quad (1.24)$$

where

$$B = \sum_{j=0}^m b_j \partial_x^j, \quad (1.25)$$

$$C = \sum_{j=0}^n c_j \partial_x^j. \quad (1.26)$$

The equation (1.24) is obtained from the compatibility condition of

$$\partial_t\psi = B\psi, \quad (1.27)$$

$$\partial_y\psi = C\psi. \quad (1.28)$$

By choosing suitable operators B and C , we can obtain several soliton equations from the equation (1.24). If we put

$$B = \partial_x^3 + \frac{3}{2}u\partial_x + v, \quad (1.29)$$

$$C = \partial_x^2 + u, \quad (1.30)$$

then the KP equation is obtained. If we suppose that u is independent of y , then the KP equation reduces to the KdV equation. M. Sato ([30, 31]) constructed the KP hierarchy and the multi-component KP hierarchy that include the KP equation. The solutions of the KP hierarchy constitute an infinite-dimensional Grassmann manifold. The unified approach to integrability makes us understand algebraically and geometrically integrable systems with infinitely many degree of freedom and their solutions. This approach is known as the Sato theory nowadays ([28]).

1.3 Relations between soliton systems and Painlevé equations

The Painlevé equations are treated in the research of the mathematical physics. It was found by T. T. Wu, B. M. McCoy, C. A. Tracy and E. Barouch ([36]) that the correlation function for the two-dimensional Ising model in the scaling region satisfies P_{III} . In the soliton theory, it was demonstrated by M. J. Ablowitz and H. Segur ([1]) that similarity solutions of the soliton equations satisfy the Painlevé equations. The relation between the isomonodromic deformation and the isospectral one was discussed; see [3, 11, 33, 34]. M. Jimbo and T. Miwa ([11]) described a procedure to reduce the isospectral deformation into the isomonodromic deformation consistently by using the τ -function. One can obtain not only the Painlevé equations themselves but also the Lax pairs of them. P_{III} and P_{IV} were obtained through the reduction from the Pöhlmeyer-Lund-Regge equation and the nonlinear Schrödinger equation, respectively. M. Noumi and Y. Yamada ([27]) introduced a Painlevé system associated with the affine root system of type $A_{n-1}^{(1)}$ including $P_{II}(A_1^{(1)})$, $P_{IV}(A_2^{(1)})$ and $P_V(A_3^{(1)})$. The systems are equivalent to similarity reductions of the n -reduced modified KP hierarchy. The coefficients of the Lax pair for the system of type $A_{n-1}^{(1)}$ are $n \times n$ matrices ([26]). The similarity reductions of the Drinfel'd-Sokolov hierarchies was investigated by T. Ikeda, S. Kakei and T. Kikuchi; see [12, 13, 14, 18]. As consequence, P_V can be obtained from the modified Yajima-Oikawa equation, and P_{VI} with four parameters can be derived from the three-wave resonant system. In the papers, [14, 18], the coefficients of the Lax pair of which they obtained were also 3×3 matrices. They showed that the 2×2 linear system can be obtained from the 3×3 linear system by the method of using the Laplace transformation ([8, 24]).

1.4 Results

We give systems of the isospectral deformations that are directly reduced to the Lax pairs for the Painlevé equations. Specially, we deal with the linear systems with 2×2 matrices, in fact the types of singular points of the linear system with 2×2 matrices correspond to the types of the Painlevé equations. Besides reductions of the anti-self-dual Yang-Mills equations to ordinary differential equations yield the Painlevé equations. The 2×2 linear system of the anti-self-dual Yang-Mills equations is also reduced to the Lax pairs for the Painlevé equations ([23]). We intend to study the Painlevé equations by relating the properties of the soliton equations to that of the Painlevé equations. In order to construct the signpost of this approach, we try to formulate the holonomic deformation by using the Sato theory.

In this paper, we consider an infinite-dimensional integrable hierarchy and give the Lax pair with 2×2 matrices for P_{VI} . This hierarchy is an extension of the two-component KP hierarchy by using additional time variables. The extension means that the hierarchy restricted to be independent of the introduced time variables is equal to the usual two-component KP hierarchy. We consider specially the $(1, 1)$ -reduction of the two-component KP hierarchy which is known as the nonlinear Schrödinger hierarchy. It is contained in the extended Zakharov-Shabat hierarchy; cf [2]. We formulate the extended hierarchy by using the Sato-Wilson formalism and then define a wave function a normal solution of the linear system. This wave function is of the form similar to the integrand of the Lauricella's hypergeometric integral.

Then we consider the holonomic deformation in the same way to the isospectral deformation. We construct a system of linear differential equations in the spectral parameter λ by using the wave function in the extended hierarchy. This linear system is of the type:

$$(1, 1, \dots, 1, \infty). \quad (1.31)$$

Here, at the infinity, $\lambda = \infty$, the Poincaré rank is considered as ∞ since the coefficient matrix $A(\lambda)$ (1.15) contains the formal Laurent series around the point $\lambda = \infty$. We obtain nonlinear systems that describe the condition of the complete integrability of the linear systems. If we reduce the type of the linear system (1.31) to the type (1.17), then the infinite-dimensional system is reduced to the Schlesinger system, from which P_{VI} is obtained.

We treat also the other Painlevé equations from the viewpoint of the usual two-component KP hierarchy. We study the nonlinear Schrödinger hierarchy

by using the Sato-Wilson formalism, and then give different wave functions, similar to the integrand of the some degenerated hypergeometric integral. The choice of the wave function can be done freely from the two-component KP hierarchy, the holonomic deformations might be dependent on it. We construct systems of linear differential equations in the spectral parameter λ by using each wave function. The linear systems thus obtained are of the types:

$$(1, 1, \infty), \quad (1.32a)$$

$$(1, \infty), \quad (1.32b)$$

$$(2, \infty), \quad (1.32c)$$

$$(\infty). \quad (1.32d)$$

We then obtain nonlinear systems that describe the condition of the complete integrability of the linear systems. If we assume the following reductions for the linear systems (1.32):

$$(1, 1, \infty) \rightarrow (1, 1, 2), \quad (1.33a)$$

$$(1, \infty) \rightarrow (1, 3), \quad (1.33b)$$

$$(2, \infty) \rightarrow (2, 2), \quad (1.33c)$$

$$(\infty) \rightarrow (4), \quad (1.33d)$$

then the infinite-dimensional systems is reduced to one-dimensional systems which yield the other Painlevé equations; see Section 4 below. It follows that the reductions of the nonlinear Schrödinger equation give rise to not only P_{IV} (see [11]), but also P_V and P_{III} .

1.5 Remarks

To study the Painlevé equations and the holonomic deformations, it is indispensable to consider the extension of the two-component KP hierarchy, which relates directly to P_{VI} . F. Nijhoff, A. Hone and N. Joshi have showed that similarity reductions of a partial differential equation of Schwarzian type (SPDE) lead to P_{VI} ([25]). They gave a Lax pair of 2×2 matrices type for the SPDE. Therefore it is quite natural to ask that there is an intimate relation between the similarity reductions of the SPDE and our result. This kind of problems remains still open. We can also obtain any deformation of a linear differential equation with rational coefficients by means of our method. Studies on such procedure are a main subject of a forthcoming paper.

2 An extension of the two-component KP hierarchy

In the present section, we study an extension of the $(1,1)$ -reduction of the two-component KP hierarchy. We give a formulation of this hierarchy by using the Sato-Wilson formalism, and then obtain an integrable system by means of the Zakharov-Shabat system.

2.1 Pseudo-differential operator

The multi-component theory of the KP hierarchy is established in the paper, [30]. The n -component KP hierarchy is formulated by matrix pseudo-differential operators of size $n \times n$, instead of scalar ones used in the one-component hierarchy. We explain some notation about the matrix pseudo-differential operators of size $n \times n$.

The action of the differential operator ∂_x on an $n \times n$ matrix $f(x)$ is

$$\partial_x f(x) = \frac{d}{dx} f(x).$$

The operator ∂_x^{-1} is defined by

$$\partial_x \partial_x^{-1} = \partial_x^{-1} \partial_x \equiv 1.$$

Pseudo-differential operators are defined by using the operators ∂_x and ∂_x^{-1} .

Definition 1. A pseudo-differential operator with matrix-coefficients of size $n \times n$ is a linear operator,

$$\mathcal{A} = \sum_m a_m(x) \partial_x^m,$$

where $a_m(x)$ is an $n \times n$ matrix-valued function of x .

A sum of pseudo-differential operators is defined in the usual way by collecting terms, and their product is defined by the following extension of Leibniz's rule,

$$\mathcal{A}\mathcal{B} = \sum_{m,n} a_m(x) \partial_x^m b_n(x) \partial_x^n = \sum_{m,n} \sum_{k=0}^{\infty} \binom{i}{k} a_m(x) b_n^{(m)}(x) \partial_x^{m+n-k},$$

where

$$\binom{i}{k} = \begin{cases} \frac{i(i-1)\dots(i-k+1)}{k!} & (k \geq 1) \\ 1 & (k = 0). \end{cases}$$

We define the differential operator part of a pseudo-differential operator \mathcal{A} by

$$(\mathcal{A})_+ = \sum_{m \geq 0} a_m(x) \partial_x^m.$$

A pseudo-differential operator possesses a unique inverse, denoted by \mathcal{A}^{-1} .

2.2 Sato Equation

In the Sato-Wilson formalism, a pseudo-differential operator called the gauge operator plays an essential role. The coefficients of the gauge operator are dependent variables in the soliton system. The condition of the isospectral deformation is given by the Sato equations that the gauge operator should satisfy.

We define the gauge operator of size 2×2 by

$$\mathcal{W} = I + \sum_{k=1}^{\infty} w_k \partial_x^{-k}, \quad (2.1)$$

whose 2×2 coefficients matrices w_k ($k \geq 1$) do not depend on the parameter x . This condition for the coefficients is equivalent to “the $(1, 1)$ -reduction”. The formal series \mathcal{W} can be inverted. Let

$$\mathcal{W}^{-1} = \sum_{k=0}^{\infty} v_k \partial_x^{-k}, \quad (2.2)$$

the first few v_k ’s are

$$v_0 = I, \quad (2.3a)$$

$$v_1 = -w_1, \quad (2.3b)$$

$$v_2 = -w_2 + w_1^2, \quad (2.3c)$$

$$v_3 = -w_3 + w_1 w_2 + w_2 w_1 - w_1^3. \quad (2.3d)$$

The gauge operator \mathcal{W} can be used to define the operator

$$\mathcal{U} = \mathcal{W} \sigma_3 \mathcal{W}^{-1} = \sigma_3 + \sum_{k=1}^{\infty} u_k \partial_x^{-k}, \quad (2.4)$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$u_k = \sum_{j=1}^k [w_j, \sigma_3] v_{k-j} \quad (k \geq 1). \quad (2.5)$$

We introduce a differential operator

$$\mathcal{S}_n = (\gamma_n I + c_n \sigma_3) \sum_{k=0}^{\infty} a_n^{-k-1} \partial_x^k \quad (n = 1, \dots, l). \quad (2.6)$$

By employing the gauge operator \mathcal{W} and the differential operator \mathcal{S}_n , we define differential operators \mathcal{B}_n ($n \geq 1$) and \mathcal{C}_n ($n = 1, \dots, l$) by

$$\mathcal{B}_n = (\mathcal{W} \sigma_3 \partial_x^n \mathcal{W}^{-1})_+ = \sum_{k=0}^{n-1} u_{n-k} \partial_x^k + \sigma_3 \partial_x^n \quad (n \geq 1), \quad (2.7)$$

$$\mathcal{C}_n = (\mathcal{W} \mathcal{S}_n \mathcal{W}^{-1})_+ = R_n \sum_{k=0}^{\infty} a_n^{-k-1} \partial_x^k \quad (n = 1, \dots, l), \quad (2.8)$$

where

$$R_n = \gamma_n I + c_n \left(\sigma_3 + \sum_{l=1}^{\infty} a_n^{-l} u_l \right) \quad (n = 1, \dots, l). \quad (2.9)$$

Matrix operators

$$W = I + \sum_{k=1}^{\infty} w_k \lambda^{-k}, \quad (2.10)$$

$$U = \sigma_3 + \sum_{k=1}^{\infty} u_k \lambda^{-k}, \quad (2.11)$$

$$S_n = (\gamma_n I + c_n \sigma_3) \sum_{k=0}^{\infty} a_n^{-k-1} \lambda^k = -\frac{\gamma_n I + c_n \sigma_3}{\lambda - a_n} \quad (n = 1, \dots, l), \quad (2.12)$$

$$B_n = \sum_{k=0}^{n-1} u_{n-k} \lambda^k + \sigma_3 \lambda^n \quad (n \geq 1), \quad (2.13)$$

$$C_n = R_n \sum_{k=0}^{\infty} a_n^{-k-1} \lambda^k = -\frac{R_n}{\lambda - a_n} \quad (n = 1, \dots, l) \quad (2.14)$$

are obtained from the pseudo-differential operators by replacing ∂_x with λ . We assume that the matrix operators satisfy

$$\partial_{t_n} W = B_n W - W \sigma_3 \lambda^n \quad (n \geq 1), \quad (2.15)$$

$$\partial_{a_n} W = C_n W - W S_n \quad (n = 1, \dots, l), \quad (2.16)$$

which we call the Sato equation hereafter.

Let us now define a wave function.

Definition 2. A wave function $\Psi(\lambda)$ is defined by the following expression:

$$\Psi(\lambda) = W \Psi_0(\lambda), \quad (2.17)$$

where

$$\begin{aligned} \Psi_0(\lambda) &= \lambda^\alpha (\lambda - 1)^\beta \prod_{n=1}^l (\lambda - a_n)^{\gamma_n} \exp(x\lambda) \\ &\times \text{diag} \left\{ \lambda^a (\lambda - 1)^b \prod_{n=1}^l (\lambda - a_n)^{c_n} \exp \left(\sum_{n=1}^{\infty} t_n \lambda^n \right), \right. \\ &\quad \left. \lambda^{-a} (\lambda - 1)^{-b} \prod_{n=1}^l (\lambda - a_n)^{-c_n} \exp \left(- \sum_{n=1}^{\infty} t_n \lambda^n \right) \right\}. \end{aligned} \quad (2.18)$$

The elements of the wave function are similar to the integrand of the Lauricella's hypergeometric integral:

$$\begin{aligned} &F_D(a, b_1, \dots, b_l, c; a_1, \dots, a_l) \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \lambda^{a-1} (1-\lambda)^{c-a-1} \prod_{n=1}^l (1 - a_n \lambda)^{-b_n} d\lambda. \end{aligned} \quad (2.19)$$

We note that the matrix-valued function $\Psi_0(\lambda)$ satisfies

$$\partial_x \Psi_0(\lambda) = \lambda \Psi_0(\lambda), \quad (2.20)$$

$$\partial_{t_n} \Psi_0(\lambda) = \sigma_3 \lambda^n \Psi_0(\lambda) = \sigma_3 \partial_x^n \Psi_0(\lambda) \quad (n \geq 1), \quad (2.21)$$

$$\partial_{a_n} \Psi_0(\lambda) = S_n \Psi_0(\lambda) = \mathcal{S}_n \Psi_0(\lambda) \quad (n = 1, \dots, l). \quad (2.22)$$

This leads to the following theorem:

Theorem 1. *If a matrix operator W satisfies the Sato equation (2.15) and (2.16), then the wave function $\Psi(\lambda)$ which can be derived from W satisfies the linear systems,*

$$\partial_x \Psi(\lambda) = \lambda \Psi(\lambda), \quad (2.23)$$

$$\partial_{t_n} \Psi(\lambda) = B_n \Psi(\lambda) \quad (n \geq 1), \quad (2.24)$$

$$\partial_{a_n} \Psi(\lambda) = C_n \Psi(\lambda) \quad (n = 1, \dots, l). \quad (2.25)$$

Proof. We have

$$\begin{aligned} \partial_x \Psi(\lambda) - \lambda \Psi(\lambda) &= \partial_x (W \Psi_0(\lambda)) - \lambda W \Psi_0(\lambda) \\ &= (\partial_x W) \Psi_0(\lambda) \\ &= 0, \end{aligned} \quad (2.26)$$

since $\partial_x W = 0$. We find

$$\begin{aligned} \partial_{t_n} \Psi(\lambda) - B_n \Psi(\lambda) &= \partial_{t_n} (W \Psi_0(\lambda)) - B_n W \Psi_0(\lambda) \\ &= (\partial_{t_n} W - B_n W + W \sigma_3 \lambda^n) \Psi_0(\lambda) \\ &= 0 \end{aligned} \quad (2.27)$$

by the Sato equation (2.15). We obtain

$$\begin{aligned} \partial_{a_n} \Psi(\lambda) - C_n \Psi(\lambda) &= \partial_{a_n} (W \Psi_0(\lambda)) - C_n W \Psi_0(\lambda) \\ &= (\partial_{a_n} W - C_n W + W S_n) \Psi_0(\lambda) \\ &= 0 \end{aligned} \quad (2.28)$$

by the Sato equation (2.16). □

The Sato equations also lead to the following theorem:

Theorem 2. *If a matrix operator W satisfies the Sato equation (2.15) and (2.16), then the matrix operators U , B_n and C_n satisfy the Lax-type systems,*

$$\partial_{t_n} U = [B_n, U] \quad (n \geq 1), \quad (2.29)$$

$$\partial_{a_n} U = [C_n, U] \quad (n = 1, \dots, l), \quad (2.30)$$

and the Zakharov-Shabat systems,

$$\partial_{t_m} B_n - \partial_{t_n} B_m + [B_n, B_m] = 0 \quad (n, m \geq 1), \quad (2.31)$$

$$\partial_{a_m} B_n - \partial_{t_n} C_m + [B_n, C_m] = 0 \quad (n \geq 1, m = 1, \dots, l), \quad (2.32)$$

$$\partial_{a_m} C_n - \partial_{a_n} C_m + [C_n, C_m] = 0 \quad (n, m = 1, \dots, l). \quad (2.33)$$

Proof. From the definition of the pseudo-differential operator \mathcal{U} (2.4), we find

$$U = W\sigma_3 W^{-1}. \quad (2.34)$$

Therefore we have

$$\begin{aligned} \partial_{t_n} U - [B_n, U] &= \partial_{t_n}(W\sigma_3 W^{-1}) - [B_n, W\sigma_3 W^{-1}] \\ &= [(\partial_{t_n} W - B_n W + W\sigma_3 \lambda^n) W^{-1}, W\sigma_3 W^{-1}] \\ &= 0 \end{aligned} \quad (2.35)$$

by the Sato equation (2.15). We obtain

$$\begin{aligned} \partial_{a_n} U - [C_n, U] &= \partial_{a_n}(W\sigma_3 W^{-1}) - [C_n, W\sigma_3 W^{-1}] \\ &= [(\partial_{a_n} W - C_n W + W S_n) W^{-1}, W\sigma_3 W^{-1}] \\ &= 0 \end{aligned} \quad (2.36)$$

by the Sato equation (2.16). We find

$$\begin{aligned} &\partial_{t_m} B_n - \partial_{t_n} B_m + [B_n, B_m] \\ &= -\partial_{t_m} \{(\partial_{t_n} W - B_n W + W\sigma_3 \lambda^n) W^{-1}\} \\ &\quad + \partial_{t_n} \{(\partial_{t_m} W - B_m W + W\sigma_3 \lambda^m) W^{-1}\} \\ &\quad - [B_n, (\partial_{t_m} W - B_m W + W\sigma_3 \lambda^m) W^{-1}] \\ &\quad - [(\partial_{t_n} W - B_n W + W\sigma_3 \lambda^n) W^{-1}, (\partial_{t_m} W + W\sigma_3 \lambda^m) W^{-1}] \\ &= 0 \end{aligned} \quad (2.37)$$

by the Sato equations (2.15). We see

$$\begin{aligned} &\partial_{a_m} B_n - \partial_{t_n} C_m + [B_n, C_m] \\ &= -\partial_{a_m} \{(\partial_{t_n} W - B_n W + W\sigma_3 \lambda^n) W^{-1}\} \\ &\quad + \partial_{t_n} \{(\partial_{a_m} W - C_m W + W S_m) W^{-1}\} \\ &\quad - [B_n, (\partial_{a_m} W - C_m W + W S_m) W^{-1}] \\ &\quad - [(\partial_{t_n} W - B_n W + W\sigma_3 \lambda^n) W^{-1}, (\partial_{a_m} W + W S_m) W^{-1}] \\ &= 0 \end{aligned} \quad (2.38)$$

by the Sato equations (2.15) and (2.16). We have

$$\begin{aligned}
& \partial_{a_m} C_n - \partial_{a_n} C_m + [C_n, C_m] \\
&= -\partial_{a_m} \{(\partial_{a_n} W - C_n W + W S_n) W^{-1}\} \\
&\quad + \partial_{a_n} \{(\partial_{a_m} W - C_m W + W S_m) W^{-1}\} \\
&\quad - [C_n, (\partial_{a_m} W - C_m W + W S_m) W^{-1}] \\
&\quad - [(\partial_{a_n} W - C_n W + W S_n) W^{-1}, (\partial_{a_m} W + W S_m) W^{-1}] \\
&= 0
\end{aligned} \tag{2.39}$$

by the Sato equation (2.16). \square

The systems (2.31) are equal to the Zakharov-Shabat systems in the $(1, 1)$ -reduction of the two-component KP hierarchy. The systems (2.32) and (2.33) are the additional ones in the extended hierarchy. So new integrable systems are obtained from the systems (2.32) and (2.33). Since the left-hand side of (2.32) is

$$\begin{aligned}
& \partial_{a_m} C_n - \partial_{a_n} C_m + [C_n, C_m] \\
&= \left(\partial_{a_n} R_m - \frac{[R_n, R_m]}{a_n - a_m} \right) \frac{1}{\lambda - a_m} - \left(\partial_{a_m} R_n - \frac{[R_n, R_m]}{a_n - a_m} \right) \frac{1}{\lambda - a_n},
\end{aligned} \tag{2.40}$$

we obtain

$$\partial_{a_n} R_m - \frac{[R_n, R_m]}{a_n - a_m} = 0. \tag{2.41}$$

Since the left-hand side of (2.32) is

$$\begin{aligned}
& \partial_{a_m} B_n - \partial_{t_n} C_m + [B_n, C_m] \\
&= \left(\partial_{t_n} R_m - \left[\sum_{l=0}^{n-1} a_m^l u_{n-l} + a_m^n \sigma_3, R_m \right] \right) \frac{1}{\lambda - a_m} \\
&\quad + \sum_{k=1}^n \left(\partial_{a_m} u_k - \left[\sum_{l=1}^{k-1} a_m^{l-1} u_{k-l} + a_m^{k-1} \sigma_3, R_m \right] \right) \lambda^{n-k},
\end{aligned} \tag{2.42}$$

we obtain systems

$$\partial_{t_n} R_m - \left[\sum_{l=1}^n a_m^{n-l} u_l + a_m^n \sigma_3, R_m \right] = 0, \quad (2.43a)$$

$$\partial_{a_m} u_k - \left[\sum_{l=1}^{k-1} a_m^{k-l-1} u_l + a_m^{k-1} \sigma_3, R_m \right] = 0 \quad (k = 1 \dots n). \quad (2.43b)$$

If we set $n = 1$, then the system (2.43) reduces to

$$\partial_{t_1} R_m - [u_1 + a_m \sigma_3, R_m] = 0, \quad (2.44a)$$

$$\partial_{a_m} u_1 - [\sigma_3, R_m] = 0. \quad (2.44b)$$

If we introduce the following parameterizations for the matrices

$$u_1 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad (2.45a)$$

$$R_m = \begin{pmatrix} g+h & e \\ f & g-h \end{pmatrix}, \quad (2.45b)$$

then we obtain a system

$$\partial_{a_m} u - 2e = 0, \quad (2.46a)$$

$$\partial_{a_m} v + 2f = 0, \quad (2.46b)$$

$$\partial_{t_1} e - 2a_m e + 2hu = 0, \quad (2.46c)$$

$$\partial_{t_1} f + 2a_m f - 2hv = 0, \quad (2.46d)$$

$$\partial_{t_1} h - fu + ev = 0, \quad (2.46e)$$

$$\partial_{t_1} g = 0. \quad (2.46f)$$

Remark 2.1. We have formulated the hierarchy by using the pseudo-differential operators. We can also formulate that by using the difference operators (see [35]). If the gauge operator \mathcal{W} does not depend on the parameter α , then we have

$$e^{\partial_\alpha} \Psi(\lambda) = \lambda \Psi(\lambda). \quad (2.47)$$

Therefore the difference operators are obtained from the pseudo-differential operators by replacing ∂_x with e^{∂_α} .

3 The extended two-component system and the sixth Painlevé equation

In this section, we consider a holonomic deformation of systems, obtained from the integrable system given in the previous section. We construct a system of linear differential equations in the spectral parameter λ by using the wave function in the extended hierarchy, and then obtain nonlinear systems that describe the condition of the complete integrability of the linear systems. We show that the infinite-dimensional system is reduced to the Schlesinger system, from which P_{VI} is obtained.

If we introduce a differential operator

$$\begin{aligned} \mathcal{V} = I & \left(\alpha - \beta \sum_{k=1}^{\infty} \partial_x^k - \sum_{n=1}^l \gamma_n \sum_{k=1}^{\infty} a_n^{-k} \partial_x^k + x \partial_x \right) \\ & + \sigma_3 \left(a - b \sum_{k=1}^{\infty} \partial_x^k - \sum_{n=1}^l c_n \sum_{k=1}^{\infty} a_n^{-k} \partial_x^k + \sum_{n=1}^{\infty} n t_n \partial_x^n \right), \end{aligned} \quad (3.1)$$

then the matrix-valued function $\Psi_0(\lambda)$ (2.18) fulfills

$$\lambda \partial_\lambda \Psi_0(\lambda) = \mathcal{V} \Psi_0(\lambda). \quad (3.2)$$

By using the gauge operator \mathcal{W} and the differential operator \mathcal{V} , we define a differential operator \mathcal{D} by

$$\mathcal{D} = (\mathcal{W} \mathcal{V} \mathcal{W}^{-1})_+ = \sum_{k=0}^{\infty} d_k \partial_x^k, \quad (3.3)$$

where

$$d_0 = \alpha I + a\sigma_3 - b \sum_{l=1}^{\infty} u_l - \sum_{n=1}^l c_n \sum_{l=1}^{\infty} a_n^{-l} u_l + \sum_{n=1}^{\infty} nt_n u_n, \quad (3.4a)$$

$$d_1 = \left(-\beta - \sum_{n=1}^l \gamma_n a_n^{-1} + x \right) I - b \left(\sigma_3 + \sum_{l=1}^{\infty} u_l \right) - \sum_{n=1}^l c_n a_n^{-1} \left(\sigma_3 + \sum_{l=1}^{\infty} a_n^{-l} u_l \right) + t_1 \sigma_3 + \sum_{n=2}^{\infty} nt_n u_{n-1}, \quad (3.4b)$$

$$d_k = \left(-\beta - \sum_{n=1}^l \gamma_n a_n^{-k} \right) I - b \left(\sigma_3 + \sum_{l=1}^{\infty} u_l \right) - \sum_{n=1}^l c_n a_n^{-k} \left(\sigma_3 + \sum_{l=1}^{\infty} a_n^{-l} u_l \right) + kt_k \sigma_3 + \sum_{n=k+1}^{\infty} nt_n u_{n-k} \quad (k \geq 2). \quad (3.4c)$$

We introduce matrix operators

$$T = \frac{\alpha I + a\sigma_3}{\lambda} + \frac{\beta I + b\sigma_3}{\lambda - 1} + \sum_{n=1}^l \frac{\gamma_n I + c_n \sigma_3}{\lambda - a_n} + \sum_{n=1}^{\infty} nt_n \sigma_3 \lambda^{n-1}, \quad (3.5)$$

$$A = \sum_{k=0}^{\infty} d_k \lambda^{k-1}. \quad (3.6)$$

We note that

$$\partial_\lambda \Psi_0(\lambda) = T \Psi_0(\lambda). \quad (3.7)$$

We assume that the matrix operator A satisfies the Sato equation with respect to the spectral parameter:

$$\partial_\lambda W = AW - WT. \quad (3.8)$$

This leads to the following theorem:

Theorem 3. *If a matrix operator W satisfies the reduction condition (3.8), then the wave function $\Psi(\lambda)$ (2.17) satisfies the linear system*

$$\partial_\lambda \Psi(\lambda) = A \Psi(\lambda). \quad (3.9)$$

Proof. We have

$$\begin{aligned}
\partial_\lambda \Psi(\lambda) - A\Psi(\lambda) &= \partial_\lambda (W\Psi_0(\lambda)) - AW\Psi_0(\lambda) \\
&= (\partial_\lambda W - AW + WT) \Psi_0(\lambda) \\
&= 0
\end{aligned} \tag{3.10}$$

by the condition (3.8). \square

The Sato equations also lead to the following theorem:

Theorem 4. *If a matrix operator W satisfies the Sato equation (2.15), (2.16) and (3.8), then the matrix operators U and A satisfy the Lax-type systems,*

$$\partial_\lambda U = [A, U], \tag{3.11}$$

and the matrix operators A , B_n and C_n satisfy the Zakharov-Shabat type systems,

$$\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] = 0 \quad (n \geq 1), \tag{3.12}$$

$$\partial_{a_n} A - \partial_\lambda C_n + [A, C_n] = 0 \quad (n = 1, \dots, l). \tag{3.13}$$

Proof. We have

$$\begin{aligned}
\partial_\lambda U - [A, U] &= \partial_\lambda (W\sigma_3 W^{-1}) - [A, W\sigma_3 W^{-1}] \\
&= [(\partial_\lambda W - AW + WT) W^{-1}, W\sigma_3 W^{-1}] \\
&= 0
\end{aligned} \tag{3.14}$$

by the Sato equation (3.8). We find

$$\begin{aligned}
&\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] \\
&= -\partial_{t_n} \{(\partial_\lambda W - AW + WT) W^{-1}\} \\
&\quad + \partial_\lambda \{(\partial_{t_n} W - B_n W + W\sigma_3 \lambda^n) W^{-1}\} \\
&\quad - [A, (\partial_{t_n} W - B_n W + W\sigma_3 \lambda^n) W^{-1}] \\
&\quad - [(\partial_\lambda W - AW + WT) W^{-1}, (\partial_{t_n} W + W\sigma_3 \lambda^n) W^{-1}] \\
&= 0
\end{aligned} \tag{3.15}$$

by the Sato equations (2.15) and (3.8). We have

$$\begin{aligned}
& \partial_{a_n} A - \partial_\lambda C_n + [A, C_n] \\
&= -\partial_{a_n} \{(\partial_\lambda W - AW + WT) W^{-1}\} \\
&\quad + \partial_\lambda \{(\partial_{a_n} W - C_n W + WS_n) W^{-1}\} \\
&\quad - [A, (\partial_{a_n} W - C_n W + WS_n) W^{-1}] \\
&\quad - [(\partial_\lambda W - AW + WT) W^{-1}, (\partial_{a_n} W + WS_n) W^{-1}] \\
&= 0
\end{aligned} \tag{3.16}$$

by the Sato equations (2.16) and (3.8). \square

If we introduce matrices

$$P = \alpha I + a\sigma_3 - b \sum_{l=1}^{\infty} u_l - \sum_{n=1}^l c_n \sum_{l=1}^{\infty} a_n^{-l} u_l + \sum_{n=1}^{\infty} n t_n u_n \tag{3.17a}$$

$$Q = \beta I + b \left(\sigma_3 + \sum_{l=1}^{\infty} u_l \right), \tag{3.17b}$$

$$T_0 = xI + t_1 \sigma_3 + \sum_{n=2}^{\infty} n t_n u_{n-1}, \tag{3.17c}$$

$$T_k = (k+1) t_{k+1} \sigma_3 + \sum_{n=k+2}^{\infty} n t_n u_{n-k-1} \quad (k \geq 1), \tag{3.17d}$$

then we have

$$A = \frac{P}{\lambda} + \frac{Q}{\lambda-1} + \sum_{n=1}^l \frac{R_n}{\lambda-a_n} + \sum_{k=0}^{\infty} T_k \lambda^k, \tag{3.18}$$

where the matrix R_n is given by (2.9). If we put $t_n \equiv 0$ ($n \geq r$), then we have $R_k \equiv 0$ ($k \geq r-1$), and A has a pole of degree r at $\lambda = \infty$. In this case, the linear system (3.9) is said to have an irregular singular point at $\lambda = \infty$ of Poincaré rank $r-1$.

By using (2.14) and (3.18), the left-hand side of the system (3.13) becomes

$$\begin{aligned}
& \partial_{a_n} A - \partial_\lambda C_n + [A, C_n] \\
&= \left(\partial_{a_n} P + \left[\frac{P}{a_n}, R_n \right] \right) \frac{1}{\lambda} + \left(\partial_{a_n} Q + \left[\frac{Q}{a_n - 1}, R_n \right] \right) \frac{1}{\lambda - 1} \\
&+ \sum_{\substack{m=1, \dots, l \\ m \neq n}} \left(\partial_{a_n} R_m + \left[\frac{R_m}{a_n - a_m}, R_n \right] \right) \frac{1}{\lambda - a_m} \\
&+ \left(\partial_{a_n} R_n - \left[\frac{P}{a_n} + \frac{Q}{a_n - 1} + \sum_{\substack{m=1, \dots, l \\ m \neq n}} \frac{R_m}{a_n - a_m} + \sum_{l=0}^{\infty} a_n {}^l T_l, R_n \right] \right) \frac{1}{\lambda - a_n} \\
&+ \sum_{k=0}^{\infty} \left(\partial_{a_n} T_k - \left[\sum_{l=k+1}^{\infty} a_n {}^{l-k-1} T_l, R_n \right] \right) \lambda^k.
\end{aligned} \tag{3.19}$$

It follows that we obtain the systems

$$\partial_{a_n} P + \left[\frac{P}{a_n}, R_n \right] = 0, \tag{3.20a}$$

$$\partial_{a_n} Q + \left[\frac{Q}{a_n - 1}, R_n \right] = 0, \tag{3.20b}$$

$$\partial_{a_n} R_m + \left[\frac{R_m}{a_n - a_m}, R_n \right] = 0 \ (m \neq n), \tag{3.20c}$$

$$\partial_{a_n} R_n - \left[\frac{P}{a_n} + \frac{Q}{a_n - 1} + \sum_{\substack{m=1, \dots, l \\ m \neq n}} \frac{R_m}{a_n - a_m} + \sum_{l=0}^{\infty} a_n {}^l T_l, R_n \right] = 0, \tag{3.20d}$$

$$\partial_{a_n} T_k - \left[\sum_{l=k+1}^{\infty} a_n {}^{l-k-1} T_l, R_n \right] = 0 \ (k \geq 0). \tag{3.20e}$$

If we put $t_n \equiv 0$ ($n \geq 1$) and $x \equiv 0$, then the coefficient matrices reduce to

$T_k \equiv 0$ ($k \geq 0$) and we have

$$\partial_{a_n} P + \left[\frac{P}{a_n}, R_n \right] = 0, \quad (3.21a)$$

$$\partial_{a_n} Q + \left[\frac{Q}{a_n - 1}, R_n \right] = 0, \quad (3.21b)$$

$$\partial_{a_n} R_m + \left[\frac{R_m}{a_n - a_m}, R_n \right] = 0 \quad (m \neq n), \quad (3.21c)$$

$$\partial_{a_n} R_n - \left[\frac{P}{a_n} + \frac{Q}{a_n - 1} + \sum_{\substack{m=1, \dots, l \\ m \neq n}} \frac{R_m}{a_n - a_m}, R_n \right] = 0. \quad (3.21d)$$

This system is nothing but the Schlesinger system ([32]). If we set $l = 1$, then we have

$$\partial_{a_1} P + \left[\frac{P}{a_1}, R_1 \right] = 0, \quad (3.22a)$$

$$\partial_{a_1} Q + \left[\frac{Q}{a_1 - 1}, R_1 \right] = 0. \quad (3.22b)$$

This system is equivalent to P_{VI} in the paper, [10].

4 The two-component KP hierarchy and the other Painlevé equations

In this section, we study holonomic deformation relating to the $(1, 1)$ -reduction of the two-component KP hierarchy. We show that systems obtained from the deformation reduces to the Painlevé equation, P_V , P_{IV} , P_{III} and P_{II} .

4.1 The fifth Painlevé equation

We explain the $(1, 1)$ -reduction of the two-component KP hierarchy. We show that the systems that describes the condition of the holonomic deformation that contains this hierarchy as a part reduces to P_V . Therefore we find

that P_V is obtained through the reduction from the nonlinear Schrödinger equation.

We define the gauge operator

$$\mathcal{W} = I + \sum_{k=1}^{\infty} w_k \partial_x^{-k} \quad (4.1)$$

whose 2×2 coefficients matrices w_k do not depend on the parameter x . This condition for the coefficients is equivalent to “the $(1, 1)$ -reduction”. By using the gauge operator \mathcal{W} , we define a pseudo-differential operator \mathcal{U} by

$$\mathcal{U} = \mathcal{W} \sigma_3 \mathcal{W}^{-1} = \sigma_3 + \sum_{k=1}^{\infty} u_k \partial_x^{-k}. \quad (4.2)$$

We define a differential operator \mathcal{B}_n by

$$\mathcal{B}_n = (\mathcal{W} \sigma_3 \partial_x^n \mathcal{W}^{-1})_+ = \sum_{k=0}^{n-1} u_{n-k} \partial_x^k + \sigma_3 \partial_x^n \quad (n \geq 1). \quad (4.3)$$

Matrix operators

$$W = I + \sum_{k=1}^{\infty} w_k \lambda^{-k}, \quad (4.4)$$

$$U = \sigma_3 + \sum_{k=1}^{\infty} u_k \lambda^{-k}, \quad (4.5)$$

$$B_n = \sum_{k=0}^{n-1} u_{n-k} \lambda^k + \sigma_3 \lambda^n \quad (n \geq 1) \quad (4.6)$$

are obtained from the pseudo-differential operators by replacing ∂_x with λ . We assume that the matrix operators satisfy the Sato equation

$$\partial_{t_n} W = B_n W - W \sigma_3 \lambda^n \quad (n \geq 1). \quad (4.7)$$

We define a wave function

$$\Psi(\lambda) = W \Psi_0(\lambda), \quad (4.8)$$

where

$$\begin{aligned} \Psi_0(\lambda) &= \lambda^\alpha (\lambda - 1)^\beta \exp(x\lambda) \\ &\times \begin{pmatrix} \lambda^a (\lambda - 1)^b \exp(\sum_{n=1}^{\infty} t_n \lambda^n) & 0 \\ 0 & \lambda^{-a} (\lambda - 1)^{-b} \exp(-\sum_{n=1}^{\infty} t_n \lambda^n) \end{pmatrix}. \end{aligned} \quad (4.9)$$

This definition of the wave function is slightly different from the usual one. The element of $\Psi_0(\lambda)$ is similar to the integrand of the integral representation of the confluent hypergeometric function:

$${}_1F_1(a; b; t) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 \lambda^{a-1} (1-\lambda)^{b-a-1} e^{t\lambda} d\lambda. \quad (4.10)$$

The difference does not affect the soliton system, but affects the system of the holonomic deformation. We note that the matrix-valued function $\Psi_0(\lambda)$ satisfies

$$\partial_x \Psi_0(\lambda) = \lambda \Psi_0(\lambda), \quad (4.11)$$

$$\partial_{t_n} \Psi_0(\lambda) = \sigma_3 \lambda^n \Psi_0(\lambda) = \sigma_3 \partial_x^n \Psi_0(\lambda) \quad (n \geq 1). \quad (4.12)$$

This leads to the following proposition:

Proposition 5. *If a matrix operator W satisfies the Sato equation (4.7), then the matrix operators U and B_n satisfy*

$$\partial_{t_n} U = [B_n, U] \quad (n \geq 1), \quad (4.13)$$

$$\partial_{t_m} B_n - \partial_{t_n} B_m + [B_n, B_m] = 0 \quad (n, m \geq 1). \quad (4.14)$$

Furthermore, the wave function $\Psi(\lambda)$ satisfies the linear systems,

$$\partial_x \Psi(\lambda) = \lambda \Psi(\lambda), \quad (4.15)$$

$$\partial_{t_n} \Psi(\lambda) = B_n \Psi(\lambda) \quad (n \geq 1). \quad (4.16)$$

If we choose $m = 1$ and $n = 2$, then the Zakharov-Shabat system (4.14)

$$\partial_{t_1} B_2 - \partial_{t_2} B_1 + [B_2, B_1] = 0 \quad (4.17)$$

yields

$$\partial_{t_1} u_1 + [u_2, \sigma_3] = 0, \quad (4.18a)$$

$$\partial_{t_1} u_2 - \partial_{t_2} u_1 + [u_2, u_1] = 0. \quad (4.18b)$$

If we use the following parameterizations for the matrices

$$u_1 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad (4.19a)$$

$$u_2 = \begin{pmatrix} -uv/2 & f \\ g & uv/2 \end{pmatrix}, \quad (4.19b)$$

then we have the nonlinear Schrödinger equation

$$\partial_{t_1} u - 2f = 0, \quad (4.20a)$$

$$\partial_{t_1} v + 2g = 0, \quad (4.20b)$$

$$\partial_{t_1} f - \partial_{t_2} u - u^2 v = 0, \quad (4.20c)$$

$$\partial_{t_1} g - \partial_{t_2} v + uv^2 = 0. \quad (4.20d)$$

We consider the holonomic deformation that contains the two-component system. If we introduce a differential operator

$$\mathcal{V} = \alpha - \beta \sum_{k=1}^{\infty} \partial_x^k + x \partial_x + \sigma_3 \left\{ a - b \sum_{k=1}^{\infty} \partial_x^k + \sum_{n=1}^{\infty} n t_n \partial_x^n \right\}, \quad (4.21)$$

then the matrix-valued function $\Psi_0(\lambda)$ (4.9) satisfies

$$\lambda \partial_\lambda \Psi_0(\lambda) = \mathcal{V} \Psi_0(\lambda). \quad (4.22)$$

By using the gauge operator \mathcal{W} and the differential operator \mathcal{V} , we define a differential operator \mathcal{D} by

$$\mathcal{D} = (\mathcal{W} \mathcal{V} \mathcal{W}^{-1})_+ = \sum_{k=0}^{\infty} d_k e^{k \partial_s}, \quad (4.23)$$

where

$$d_0 = \alpha I + a \sigma_3 - b \sum_{l=1}^{\infty} u_l + \sum_{n=1}^{\infty} n t_n u_n, \quad (4.24a)$$

$$d_1 = (-\beta + x) I - b \left(\sigma_3 + \sum_{l=1}^{\infty} u_l \right) + t_1 \sigma_3 + \sum_{n=2}^{\infty} n t_n u_{n-1}, \quad (4.24b)$$

$$d_k = -\beta I - b \left(\sigma_3 + \sum_{l=1}^{\infty} u_l \right) + k t_k \sigma_3 + \sum_{n=k+1}^{\infty} n t_n u_{n-k} \quad (k \geq 2). \quad (4.24c)$$

We introduce matrix operators

$$T = \frac{\alpha I + a\sigma_3}{\lambda} + \frac{\beta I + b\sigma_3}{\lambda - 1} + \sum_{n=1}^{\infty} nt_n\sigma_3\lambda^{n-1}, \quad (4.25)$$

$$A = \sum_{k=0}^{\infty} d_k\lambda^{k-1}. \quad (4.26)$$

We note that

$$\partial_\lambda \Psi_0(\lambda) = T\Psi_0(\lambda). \quad (4.27)$$

We assume that the matrix operator A satisfies the condition

$$\partial_\lambda W = AW - WT. \quad (4.28)$$

This leads to the following proposition:

Proposition 6. *If a matrix operator W satisfies the reduction condition (4.28), then the matrix operators U , A and B_n satisfy*

$$\partial_\lambda U = [A, U], \quad (4.29)$$

$$\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] = 0 \quad (n \geq 1). \quad (4.30)$$

Furthermore, the wave function $\Psi(\lambda)$ (4.8) satisfies the linear system,

$$\partial_\lambda \Psi(\lambda) = A\Psi(\lambda). \quad (4.31)$$

If we introduce matrices

$$P = \alpha I + a\sigma_3 - b \sum_{l=1}^{\infty} u_l + \sum_{n=1}^{\infty} nt_n u_n, \quad (4.32a)$$

$$Q = \beta I + b \left(\sigma_3 + \sum_{l=1}^{\infty} u_l \right), \quad (4.32b)$$

$$T_0 = xI + t_1\sigma_3 + \sum_{n=2}^{\infty} nt_n u_{n-1}, \quad (4.32c)$$

$$T_k = (k+1)t_{k+1}\sigma_3 + \sum_{n=k+2}^{\infty} nt_n u_{n-k-1} \quad (k \geq 1), \quad (4.32d)$$

then we have

$$A = \frac{P}{\lambda} + \frac{Q}{\lambda - 1} + \sum_{k=0}^{\infty} T_k \lambda^k. \quad (4.33)$$

By using (4.6) and (4.33), the left-hand side of the system (4.30) with $n = 1$ turns

$$\begin{aligned}
& \partial_{t_1} A - \partial_\lambda B_1 + [A, B_1] \\
&= (\partial_{t_1} P + [P, u_1]) \frac{1}{\lambda} + (\partial_{t_1} Q + [Q, u_1 + \sigma_3]) \frac{1}{\lambda - 1} \\
&+ \partial_{t_1} T_0 - \sigma_3 + [T_0, u_1] + [P + Q, \sigma_3] \\
&+ \sum_{k=1}^{\infty} (\partial_{t_1} T_k + [T_k, u_1] + [T_{k-1}, \sigma_3]) \lambda^k.
\end{aligned} \tag{4.34}$$

Therefore we obtain the systems

$$\partial_{t_1} P + [P, u_1] = 0, \tag{4.35a}$$

$$\partial_{t_1} Q + [Q, u_1 + \sigma_3] = 0, \tag{4.35b}$$

$$\partial_{t_1} T_0 - \sigma_3 + [T_0, u_1] + [P + Q, \sigma_3] = 0, \tag{4.35c}$$

$$\partial_{t_1} T_k + [T_k, u_1] + [T_{k-1}, \sigma_3] = 0 \quad (k \geq 1). \tag{4.35d}$$

If we put $t_n \equiv 0$ ($n \geq 2$), then the coefficient matrices reduce to $T_0 = t_1 \sigma_3$, $T_k \equiv 0$ ($k \geq 1$), and then we have

$$\partial_{t_1} P + [P, u_1] = 0, \tag{4.36a}$$

$$\partial_{t_1} Q + [Q, u_1 + \sigma_3] = 0. \tag{4.36b}$$

This systems is equivalent to P_V in the paper, [10].

Remark 4.1. We can also formulate this hierarchy by using the difference operators ([35]). If the gauge operator \mathcal{W} do not depend on the parameter α , then we have

$$e^{\partial_\alpha} \Psi(\lambda) = \lambda \Psi(\lambda). \tag{4.37}$$

So the difference operators are obtained from the pseudo-differential operators by replacing ∂_x with e^{∂_α} .

4.2 The fourth Painlevé equation

We consider the different holonomic deformation that relates to the hierarchy in the previous subsection. We show that the system that describes the deformation condition reduces to P_{IV} . This fact follows the result in the paper, [11].

We employ the same soliton system in the previous subsection. But we define the wave function as follows:

$$\Psi(\lambda) = W\Psi_0(\lambda), \quad (4.38)$$

where

$$\Psi_0(\lambda) = \lambda^\alpha \exp(x\lambda) \begin{pmatrix} \lambda^a \exp(\sum_{n=1}^{\infty} t_n \lambda^n) & 0 \\ 0 & \lambda^{-a} \exp(-\sum_{n=1}^{\infty} t_n \lambda^n) \end{pmatrix}. \quad (4.39)$$

The element of $\Psi_0(\lambda)$ is similar to the integrand of the integral representation of the Hermite-Weber function:

$$H_\nu(t) = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \lambda^{-\nu-1} e^{2t\lambda-\lambda^2} d\lambda. \quad (4.40)$$

The matrix-valued function $\Psi_0(\lambda)$ satisfies

$$\partial_x \Psi_0(\lambda) = \lambda \Psi_0(\lambda), \quad (4.41)$$

$$\partial_{t_n} \Psi_0(\lambda) = \sigma_3 \lambda^n \Psi_0(\lambda) = \sigma_3 \partial_x^n \Psi_0(\lambda) \quad (n \geq 1). \quad (4.42)$$

This leads to the following proposition:

Proposition 7. *If a matrix operator W satisfies the Sato equation (4.7), then the wave function $\Psi(\lambda)$ satisfies the linear systems,*

$$\partial_x \Psi(\lambda) = \lambda \Psi(\lambda), \quad (4.43)$$

$$\partial_{t_n} \Psi(\lambda) = B_n \Psi(\lambda) \quad (n \geq 1). \quad (4.44)$$

We present the reduction condition for the soliton system. If we introduce a differential operator

$$\mathcal{T} = I(\alpha + x\partial_x) + \sigma_3 \left(a + \sum_{n=1}^{\infty} n t_n \partial_x^n \right), \quad (4.45)$$

then the matrix-valued function $\Psi_0(\lambda)$ (4.39) satisfies

$$\lambda \partial_\lambda \Psi_0(\lambda) = \mathcal{T} \Psi_0(\lambda). \quad (4.46)$$

By using the gauge operator \mathcal{W} and the differential operator \mathcal{T} , we define a differential operator \mathcal{A} by

$$\mathcal{A} = (\mathcal{W}\mathcal{T}\mathcal{W}^{-1})_+ = \sum_{k=0}^{\infty} a_k \partial_x^k, \quad (4.47)$$

where

$$a_0 = \alpha I + a\sigma_3 + \sum_{n=1}^{\infty} nt_n u_n, \quad (4.48a)$$

$$a_1 = xI + t_1\sigma_3 + \sum_{n=2}^{\infty} nt_n u_{n-1}, \quad (4.48b)$$

$$a_k = kt_k\sigma_3 + \sum_{n=k+1}^{\infty} nt_n u_{n-k} \quad (k \geq 2). \quad (4.48c)$$

We introduce matrix operators

$$T = I(\alpha + x\lambda) + \sigma_3 \left(a + \sum_{n=1}^{\infty} nt_n \lambda^n \right), \quad (4.49)$$

$$A = \sum_{k=0}^{\infty} a_k \lambda^k. \quad (4.50)$$

We assume that the matrix operator A satisfies

$$\lambda \partial_\lambda W = AW - WT. \quad (4.51)$$

This leads to the following proposition:

Proposition 8. *If a matrix operator W satisfies the reduction condition (4.51), then the matrix operators U , A and B_n satisfy*

$$\lambda \partial_\lambda U = [A, U], \quad (4.52)$$

$$\partial_{t_n} A - \lambda \partial_\lambda B_n + [A, B_n] = 0 \quad (n \geq 1). \quad (4.53)$$

Furthermore, the wave function $\Psi(\lambda)$ (4.38) satisfies the linear system,

$$\lambda \partial_\lambda \Psi(\lambda) = A \Psi(\lambda). \quad (4.54)$$

Remark 4.2. If we put $t_n \equiv 0$ ($n \geq l$), then we have $a_k \equiv 0$ ($k \geq l$). In this case, the linear system (4.54) has a regular singular point at $\lambda = 0$ and an irregular singular point at $\lambda = \infty$ of Poincaré rank $l - 1$. Hence we guess that the systems (4.53) are equivalent to the fourth Painlevé equation with several variables; see [15, 16, 17].

By using (4.6) and (4.50), the left-hand side of the system (4.53) with $n = 1$ turns

$$\begin{aligned} & \partial_{t_1} A - \lambda \partial_\lambda B_1 + [A, B_1] \\ &= \partial_{t_1} a_0 + [a_0, u_1] + (\partial_{t_1} a_1 - \sigma_3 + [a_1, u_1] + [a_0, \sigma_3]) \lambda \\ &+ \sum_{k=2}^{\infty} (\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3]) \lambda^k. \end{aligned} \quad (4.55)$$

Hence we have the systems

$$\partial_{t_1} a_0 + [a_0, u_1] = 0, \quad (4.56a)$$

$$\partial_{t_1} a_1 - \sigma_3 + [a_1, u_1] + [a_0, \sigma_3] = 0, \quad (4.56b)$$

$$\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3] = 0 \quad (k \geq 2). \quad (4.56c)$$

If we put $t_2 \equiv 1/2$, $t_n \equiv 0$ ($n \geq 3$), then the coefficient matrices reduce to $a_2 = \sigma_3$, $a_k \equiv 0$ ($k \geq 3$), and we have

$$\partial_{t_1} a_0 + [a_0, u_1] = 0, \quad (4.57a)$$

$$\partial_{t_1} a_1 - \sigma_3 + [a_1, u_1] + [a_0, \sigma_3] = 0. \quad (4.57b)$$

This systems is equivalent to P_{IV} in the paper, [10].

4.3 The third Painlevé equation

We present that the system that is the condition of the different holonomic deformation reduces to P_{III} . So we find that P_{III} is obtained through the reduction from the nonlinear Schrödinger equation.

We employ the same soliton system in the previous subsection, and we give another reduction condition for the soliton system. If we introduce a differential operator

$$\mathcal{T} = I \left(\alpha \partial_x + x \partial_x^2 \right) + \sigma_3 \left(a \partial_x + \sum_{n=1}^{\infty} n t_n \partial_x^{n+1} \right), \quad (4.58)$$

then the matrix-valued function $\Psi_0(\lambda)$ (4.39) satisfies

$$\lambda^2 \partial_\lambda \Psi_0(\lambda) = \mathcal{T} \Psi_0(\lambda). \quad (4.59)$$

By using the gauge operator \mathcal{W} and the differential operator \mathcal{T} , we define a differential operator \mathcal{A} by

$$\mathcal{A} = (\mathcal{W} \mathcal{T} \mathcal{W}^{-1})_+ = \sum_{k=0}^{\infty} a_k \partial_x^k, \quad (4.60)$$

where

$$a_0 = -w_1 + au_1 + \sum_{n=1}^{\infty} nt_n u_{n+1}, \quad (4.61a)$$

$$a_1 = \alpha I + a\sigma_3 + \sum_{n=1}^{\infty} nt_n u_n, \quad (4.61b)$$

$$a_2 = xI + t_1 \sigma_3 + \sum_{n=2}^{\infty} nt_n u_{n-1}, \quad (4.61c)$$

$$a_k = (k-1)t_{k-1} \sigma_3 + \sum_{n=k}^{\infty} nt_n u_{n-k+1} \quad (k \geq 3). \quad (4.61d)$$

We introduce matrix operators

$$T = I(\alpha\lambda + x\lambda^2) + \sigma_3 \left(a\lambda + \sum_{n=1}^{\infty} nt_n \lambda^{n+1} \right), \quad (4.62)$$

$$A = \sum_{k=0}^{\infty} a_k \lambda^k. \quad (4.63)$$

We assume that the matrix operator A satisfies

$$\lambda^2 \partial_\lambda W = AW - WT. \quad (4.64)$$

This leads to the following proposition:

Proposition 9. *If a matrix operator W satisfies the reduction condition (4.64), then the matrix operators U , A and B_n satisfy*

$$\lambda^2 \partial_\lambda U = [A, U], \quad (4.65)$$

$$\partial_{t_n} A - \lambda^2 \partial_\lambda B_n + [A, B_n] = 0 \quad (n \geq 1). \quad (4.66)$$

Furthermore, the wave function $\Psi(\lambda)$ (4.38) satisfies the linear system,

$$\lambda^2 \partial_\lambda \Psi(\lambda) = A \Psi(\lambda). \quad (4.67)$$

By using (4.6) and (4.63), the left-hand side of the system (4.66) with $n = 1$ is

$$\begin{aligned} & \partial_{t_1} A - \lambda^2 \partial_\lambda B_1 + [A, B_1] \\ &= \partial_{t_1} a_0 + [a_0, u_1] + (\partial_{t_1} a_1 + [a_1, u_1] + [a_0, \sigma_3]) \lambda \\ & \quad + (\partial_{t_1} a_2 - \sigma_3 + [a_2, u_1] + [a_1, \sigma_3]) \lambda^2 \\ & \quad + \sum_{k=3}^{\infty} (\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3]) \lambda^k. \end{aligned} \quad (4.68)$$

Thus we obtain the systems

$$\partial_{t_1} a_0 + [a_0, u_1] = 0, \quad (4.69a)$$

$$\partial_{t_1} a_1 + [a_1, u_1] + [a_0, \sigma_3] = 0, \quad (4.69b)$$

$$\partial_{t_1} a_2 - \sigma_3 + [a_2, u_1] + [a_1, \sigma_3] = 0, \quad (4.69c)$$

$$\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3] = 0 \quad (k \geq 3). \quad (4.69d)$$

If we put $t_n \equiv 0$ ($n \geq 2$), then the coefficient matrices reduce to $a_2 = t_1 \sigma_3$, $a_k \equiv 0$ ($k \geq 3$), and then we have

$$\partial_{t_1} a_0 + [a_0, u_1] = 0, \quad (4.70a)$$

$$\partial_{t_1} a_1 + [a_1, u_1] + [a_0, \sigma_3] = 0. \quad (4.70b)$$

We can obtain P_{III} from this system (4.70).

4.4 The second Painlevé equation

We present that the system that describes the condition of the different holonomic deformation reduces to P_{II} .

We employ the same soliton system in Subsection 4.1. However we define the wave function as follows:

$$\Psi(\lambda) = W \Psi_0(\lambda), \quad (4.71)$$

where

$$\Psi_0(\lambda) = \lambda^\alpha e^{x\lambda} \begin{pmatrix} \exp(\sum_{n=1}^{\infty} t_n \lambda^n) & 0 \\ 0 & \exp(-\sum_{n=1}^{\infty} t_n \lambda^n) \end{pmatrix}. \quad (4.72)$$

Needless to say, the matrix-valued function $\Psi_0(\lambda)$ satisfies

$$\partial_x \Psi_0(\lambda) = \lambda \Psi_0(\lambda), \quad (4.73)$$

$$\partial_{t_n} \Psi_0(\lambda) = \sigma_3 \lambda^n \Psi_0(\lambda) = \sigma_3 \partial_x^n \Psi_0(\lambda) \quad (n \geq 1). \quad (4.74)$$

This leads to the following proposition:

Proposition 10. *If a matrix operator W satisfies the Sato equation (4.7), then the wave function $\Psi(\lambda)$ satisfies the linear systems,*

$$\partial_x \Psi(\lambda) = \lambda \Psi(\lambda), \quad (4.75)$$

$$\partial_{t_n} \Psi(\lambda) = B_n \Psi(\lambda) \quad (n \geq 1). \quad (4.76)$$

We give the reduction condition for the soliton system. If we introduce a differential operator

$$\mathcal{T} = I (\alpha \partial_x^{-1} + x) + \sigma_3 \sum_{n=1}^{\infty} n t_n \partial_x^{n-1}, \quad (4.77)$$

then the matrix-valued function $\Psi_0(\lambda)$ (4.72) satisfies

$$\partial_\lambda \Psi_0(\lambda) = \mathcal{T} \Psi_0(\lambda). \quad (4.78)$$

By using the gauge operator \mathcal{W} and the differential operator \mathcal{T} , we define a differential operator \mathcal{A} by

$$\mathcal{A} = (\mathcal{W} \mathcal{T} \mathcal{W}^{-1})_+ = \sum_{k=0}^{\infty} a_k e^{k \partial_s}, \quad (4.79)$$

where

$$a_0 = xI + t_1 \sigma_3 + \sum_{n=2}^{\infty} n t_n u_{n-1}, \quad (4.80)$$

$$a_k = (k+1) t_{k+1} \sigma_3 + \sum_{n=k+2}^{\infty} n t_n u_{n-k-1} \quad (k \geq 1). \quad (4.81)$$

We introduce matrix operators

$$T = I(\alpha\lambda^{-1} + x) + \sigma_3 \sum_{n=1}^{\infty} nt_n \lambda^{n-1}, \quad (4.82)$$

$$A = \sum_{k=0}^{\infty} a_k \lambda^k. \quad (4.83)$$

We assume that the matrix operator A satisfies

$$\partial_\lambda W = AW - WT. \quad (4.84)$$

This leads to the following proposition:

Proposition 11. *If a matrix operator W satisfies the reduction condition (4.84), then the matrix operators U , A and B_n satisfy*

$$\partial_\lambda U = [A, U], \quad (4.85)$$

$$\partial_{t_n} A - \partial_\lambda B_n + [A, B_n] = 0 \quad (n \geq 1). \quad (4.86)$$

Furthermore, the wave function $\Psi(\lambda)$ (4.71) satisfies the linear system,

$$\partial_\lambda \Psi(\lambda) = A\Psi(\lambda). \quad (4.87)$$

Remark 4.3. If we put $t_n \equiv 0$ ($n \geq l$), then we have $a_k \equiv 0$ ($k \geq l - 1$). In this case, the linear system (4.87) has an irregular singular point at $\lambda = \infty$ of Poincaré rank $l - 1$. So we guess that the systems (4.86) are equivalent to the A_g -system; see [20, 21, 22].

By using (4.6) and (4.83), the left-hand side of the system (4.86) with $n = 1$ turns

$$\begin{aligned} & \partial_{t_1} A - \partial_\lambda B_1 + [A, B_1] \\ &= \partial_{t_1} a_0 - \sigma_3 + [a_0, u_1] + \sum_{k=1}^{\infty} (\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3]) \lambda^k. \end{aligned} \quad (4.88)$$

So we have the systems

$$\partial_{t_1} a_0 - \sigma_3 + [a_0, u_1] = 0, \quad (4.89a)$$

$$\partial_{t_1} a_k + [a_k, u_1] + [a_{k-1}, \sigma_3] = 0 \quad (k \geq 1). \quad (4.89b)$$

If we put $t_3 \equiv 1/3$, $t_n \equiv 0$ ($n = 2, n \geq 4$), then the coefficient matrices reduce to $a_2 = \sigma_3$, $a_k \equiv 0$ ($k \geq 3$), and we have

$$\partial_{t_1} a_0 - \sigma_3 + [a_0, u_1] = 0, \quad (4.90a)$$

$$\partial_{t_1} a_1 + [a_1, u_1] + [a_0, \sigma_3] = 0. \quad (4.90b)$$

This systems is equivalent to P_{II} in the paper, [10].

Acknowledgments

The author wishes to thank Professor Michio Jimbo, Professor Saburo Kakei, Professor Masatoshi Noumi, Professor Yousuke Ohyama, Professor Kazuo Okamoto, Professor Hidetaka Sakai, Professor Tetsuji Tokihiro, Doctor Teruhisa Tsuda and Professor Ralph Willox for valuable comments. This work is partially supported by the Japan Society for the Promotion of Science (JSPS).

References

- [1] M. J. Ablowitz and H. Segur, Exact linearization of Painlevé transcendent, *Phys. Rev. Lett.* **38** (1977), 1103–1106.
- [2] L. A. Dickey, Isomonodromic deformations and integrable systems, *Lett. Math. Phys.* **54** (2000), 165–179.
- [3] H. Flaschka and A. C. Newell, Monodromy- and spectrum-preserving deformations. I, *Comm. Math. Phys.* **76** (1980), 65–116.
- [4] R. Fuchs, Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen, *Math. Ann.* **63** (1907), 301–321.
- [5] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes, *Acta. Math.* **33** (1910), 1–55.
- [6] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Method for solving the Korteweg-de Vries equation, *Phys. Rev. Lett.* **19** (1967), 1095–1097.

- [7] R. Garnier, Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, *Ann. Sci. École Norm. Sup.* **29** (1912), 1–126.
- [8] J. Harnad, Dual isomonodromic deformations and moment maps to loop algebras, *Comm. Math. Phys.* **166** (1994), 337–365.
- [9] M. Jimbo, T. Miwa and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I, *Physica D* **2** (1981), 306–352.
- [10] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II, *Physica D* **2** (1981), 407–448.
- [11] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III, *Physica D* **4** (1981), 26–46.
- [12] S. Kakei and T. Kikuchi, Affine Lie group approach to a derivative nonlinear Schrödinger equation and its similarity reduction, *Int. Math. Res. Not.* **78** (2004), 4181–4209.
- [13] S. Kakei and T. Kikuchi, Solutions of a derivative nonlinear Schrödinger hierarchy and its similarity reduction, *Glasgow Math. J.* **47** (2005), 99–107.
- [14] S. Kakei and T. Kikuchi, The sixth Painlevé equation as similarity reduction of $\widehat{\mathfrak{gl}}_3$ generalized Drinfel'd-Sokolov hierarchy, *Lett. Math. Phys.* **79** (2007), 221–234.
- [15] H. Kawamuko, On the holonomic deformation of linear differential equations, *Proc. Japan Acad. Ser. A Math. Sci.* **73** (1997), 152–154.
- [16] H. Kawamuko, On the polynomial Hamiltonian structure associated with the $L(1, g + 2; g)$ type, *Proc. Japan Acad. Ser. A Math. Sci.* **73** (1997), 155–157.
- [17] H. Kawamuko, On the holonomic deformation of linear differential equations with a regular singular point and an irregular singular point, *Kyushu J. Math.* **57** (2003), 1–28.

- [18] T. Kikuchi, T. Ikeda and S. Kakei, Similarity reduction of the modified Yajima-Oikawa equation, *J. Phys. A: Math. Gen.* **36** (2003), 11465–11480.
- [19] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968), 467–490.
- [20] D. Liu, On the holonomic deformation of linear differential equations of A_4 type, *Kyushu J. Math* **51** (2001), 393–412.
- [21] D. Liu, Holonomic deformation of linear differential equations of the A_3 type, *J. Math. Sci. Univ. Tokyo* **5** (1998), 435–458.
- [22] D. Liu, On the holonomic deformation of linear differential equations of the A_g type, *J. Math. Sci. Univ. Tokyo* **8** (2001), 559–594.
- [23] L. J. Mason and N. M. J. Woodhouse, Self-duality and the Painlevé transcendents, *Nonlinearity* **6** (1993), 569–581.
- [24] M. Mazzocco, Painlevé sixth equation as isomonodromic deformations equation of an irregular system, *The Kowalevski property (Leeds, 2000)*, 219–238, CRM Proc. Lecture Notes, 32, *Amer. Math. Soc., Providence, RI*, 2002.
- [25] F. Nijhoff, A. Hone and N. Joshi, On a Schwarzian PDE associated with the KdV hierarchy, *Phys. Lett. A* **267** (2000), 147–156.
- [26] M. Noumi, Affine Weyl group approach to Painlevé equations, *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, 497–509, *Higher Ed. Press, Beijing*, 2002.
- [27] M. Noumi and Y. Yamada, Higher order Painlevé equations of type $A_l^{(1)}$, *Funkcial. Ekvac.* **41** (1998), 483–503.
- [28] Y. Ohta, J. Satsuma, D. Takahashi and T. Tokihiro, An elementary introduction to Sato theory, *Progr. Theoret. Phys. Suppl.* **94** (1988), 210–241.
- [29] P. Painlevé, Sur les équations différentielles du second ordre à points critiques fixes, *Comptes Rendus Acad. Sci. Paris* **143** (1906), 1111–1117.

- [30] M. Sato, Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold, *Sūrikaisekikenkyūsho kōkyūroku* **439** (1981), 30–46.
- [31] M. Sato and Y. Sato, Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold, *Nonlinear partial differential equations in applied science (Tokyo, 1982)*, 259–271, North-Holland Math. Stud., 81, North-Holland, Amsterdam, 1983.
- [32] L. Schlesinger, Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten, *J. für Math.* **141** (1912), 96–145.
- [33] K. Ueno, Monodromy preserving deformation and its application to soliton theory, *Proc. Japan Acad. Ser. A Math. Sci.* **56** (1980), 103–108.
- [34] K. Ueno, Monodromy preserving deformation and its application to soliton theory. II, *Proc. Japan Acad. Ser. A Math. Sci.* **56** (1980), 210–215.
- [35] K. Ueno and K. Takasaki, Toda lattice hierarchy, *Group Representations and Systems of Differential Equations (Tokyo, 1982)*, 1–95, Adv. Stud. Pure Math., 4, North-Holland, Amsterdam, 1984.
- [36] T. T. Wu, B. M. McCoy, C. A. Tracy and E. Barouch, Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region, *Phys. Rev. B* **13** (1976), 316–374.
- [37] N. J. Zabusky and M. D. Kruskal, Interaction of “solitons” in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.* **15** (1965), 240–243.
- [38] V. E. Zakharov and A. B. Shabat, A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I, *Functional Anal. Appl.* **8** (1974), 226–235.